Norwegian University of Science and Technology
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TMA4195 Mathematical Modelling Autumn 2019

Exercise set 2

1 Let

$$u(x) = \mathrm{e}^{-10x} + \mathrm{e}^{-100x} \quad \text{for} \quad x \in [0,1]. \tag{1}$$

Suggest (natural) scales for x and indicate where in $[0,1]$ their use is reasonable.

2 (Problem 7 p. 32 in Logan)
A rocket blasts off from the earth's surface. During the initial phase of flight, fuel is burned at the maximum possible rate α, and the exhaust gas is expelled downward with velocity β relative to the velocity of the rocket. The motion is governed by the following set of equations:

$$m'(t) = -\alpha, \qquad m(0) = M, \tag{2}$$

$$x''(t) = \frac{\alpha\beta}{m(t)} - \frac{g}{\left(1 + \frac{x(t)}{R}\right)^2}, \qquad x(0) = x'(0) = 0, \tag{3}$$

where $m(t)$ is the mass of the rocket, $x(t)$ is the height above the earth's surface, M is the initial mass, g is the gravitational constant, and R is the radius of the earth. Reformulate the problem in terms of dimensionless variables using appropriate scales for m, x, t.
(*Hint*: Scale m and x by obvious choices, then choose the time scale by balancing equation (3); assume that the acceleration is due primarily to fuel burning and that the gravitational force is small in comparison.)

3 (Problem 4.2.6 p. 55 in Krogstad)
Consider the problem

$$\begin{gathered} y''(t) + \varepsilon y'(t) + 1 = 0, \\ y(0) = 0, \qquad y'(0) = 0, \qquad 0 < \varepsilon \ll 1. \end{gathered} \tag{4}$$

Determine the start of the perturbation expansion $y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$ to the solution for $t \geq 0$. Compare to the exact solution. (Hint: The general solution of (4) has the form $y(t) = A + Be^{-\varepsilon t} - \frac{t}{\varepsilon}$)

4 (Problem 4.2.7 p. 55 in Krogstad)
This problem is somewhat similar to the sinking object in a fluid that has been discussed in the lecture. However, we ignore now gravity (or assume that we are in a situation where it is much smaller than the other forces) but instead assume that the friction is non-linear. In this case, a possible model for the velocity reads as

$$m\frac{dv^*}{dt^*} = -av^* + b(v^*)^2, \tag{5}$$

with initial velocity

$$v^*(0) = V_0. \tag{6}$$

Here told that a, $b > 0$, and we assume that $bV_0 \ll a$ (that is, the linear part of the friction is dominant).

(a) Find the (obvious) scale for v^* and then the scale for time, T, from the simplified equation $m\frac{dv^*}{dt^*} = -av^*$ and the "rule of thumb"

$$T = \frac{\max|v^*|}{\max|dv^*/dt^*|}.$$

Show that this scaling leads to the equation

$$\frac{dv}{dt} = -v + \varepsilon v^2,\ v(0) = 1,\ \varepsilon \ll 1. \tag{7}$$

(b) Determine v_0 and v_1 of the series expansion $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$. Is this result reasonable for all $t > 0$?
Note that the general solution of $\dot{y} = -y + \varepsilon y^2 = 0$ is

$$y(t) = \frac{e^{-t}}{C + \varepsilon e^{-t}},$$

where C is a constant.